

Closing Thu: TN 3

Closing Tue: TN 4 (sigma notation practice)

Closing next Thu: TN 5 (last assignment)

Final: Sat, June 2<sup>nd</sup>, 5:00-7:50pm, KANE 130

**Taylor's Inequality (error bound):**

On a given interval  $[a,b]$ ,

if  $|f^{(n+1)}(x)| \leq M$ , then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - b|^{n+1}$$

**Recall:**

$$T_n(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

$$= \frac{1}{0!} \underbrace{f(b)}_0 + \frac{1}{1!} \underbrace{f'(b)}_1 (x-b) + \frac{1}{2!} \underbrace{f''(b)}_0 (x-b)^2 + \dots + \frac{1}{n!} f^{(n)}(b)(x-b)^n$$

stop! ↓

**Entry Task:**

Find the 7<sup>th</sup> Taylor polynomial for  $f(x) = \sin(x)$ , based at  $b = 0$ .

Find a bound on the error over the interval  $[-3,3]$ .

$$\sin(x) \approx 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7$$

$$T_7(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

- $f(x) = \sin(x)$       $f(0) = 0$
  - $f'(x) = \cos(x)$       $f'(0) = 1$
  - $f''(x) = -\sin(x)$     $f''(0) = 0$
  - $f'''(x) = -\cos(x)$     $f'''(0) = -1$
  - $f^{(4)}(x) = \sin(x)$     $f^{(4)}(0) = 0$
- 2 REPEAT

Error Bound:  $|f^{(8)}(x)| = |\sin(x)| \leq 1$

$$\text{Error} \leq \frac{1}{8!} |x|^8 \leq \frac{1}{8!} 3^8 \approx 0.1627$$

$$\sin(x) \approx T_7(x) \pm 0.1627$$

on  $[-2, 2]$

## TN 4: Taylor Series

Def'n:

The **Taylor Series** for  $f(x)$  based at  $b$  is

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n \rightarrow \infty} T_n(x)$$

If the limit exists at a particular  $x$ , then we say the series **converges** at  $x$ . Otherwise, we say it **diverges** at  $x$ .

The **open interval of convergence** is the largest open interval of values over which the series converges.

Note:

If

$$\lim_{n \rightarrow \infty} \frac{M}{(n+1)!} |x-b|^{n+1} = 0$$

then the error goes to zero and  $x$  is in the open interval of convergence.

NOTE for  $\sin(x)$  on  $[-3, 3]$   
the error bound would look like

$$\frac{1}{(n+1)!} 3^{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Ex)

$$\frac{3^{100}}{100!} = \frac{3 \cdot 3 \cdots 3}{100 \cdot 99 \cdots 3 \cdot 2} \leftarrow \text{MUCH, MUCH LARGER}$$

A few patterns we now know:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots \Rightarrow e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \Rightarrow \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \Rightarrow \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

These three converge for ALL values of  $x$ . So the **open interval of convergence** for each series above is  $(-\infty, \infty)$

NOTE:

	$k=0$	$k=1$	$k=2$	$k=3$	
$(-1)^k$	+	-	+	-	... ALTERNATING SIGN
$2k+1$	1	3	5	7	... ODDS
$2k$	0	2	4	6	... EVENS

AWESOME!

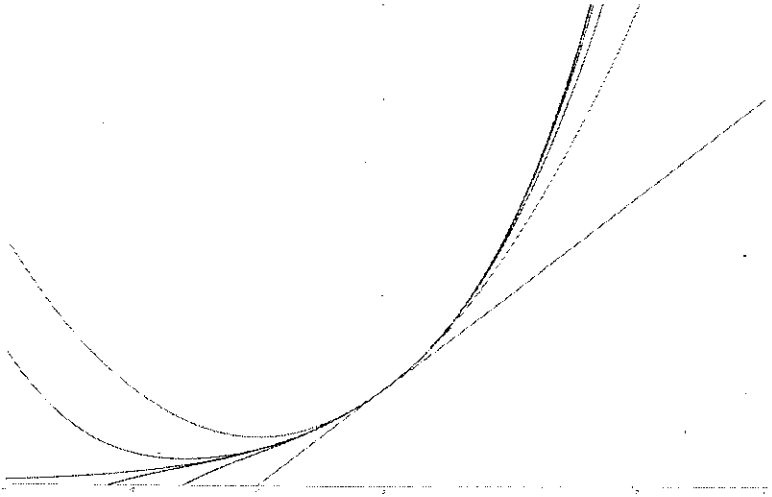
ASIDE

$$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

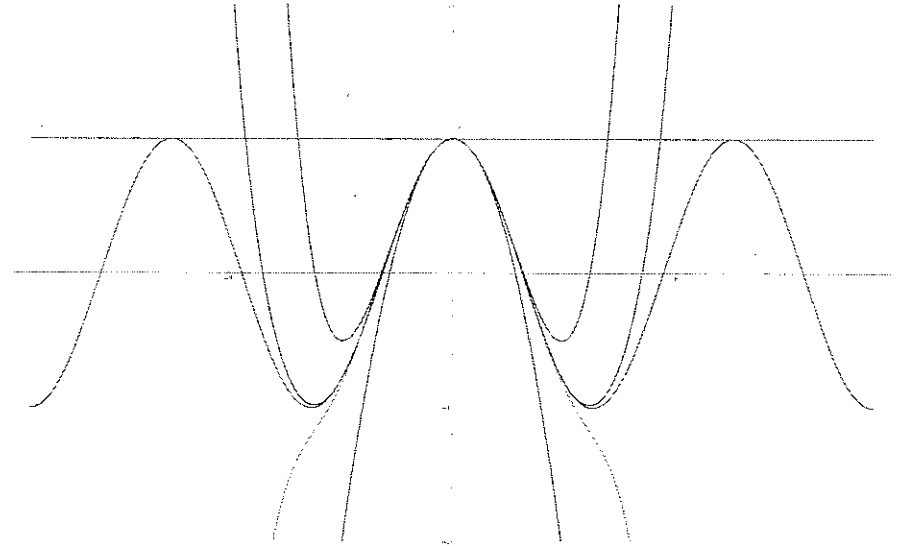
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{4}\right)^{2k+1} = \frac{\sqrt{2}}{2} \text{ COOL!}$$

Visuals of Taylor Polynomials:

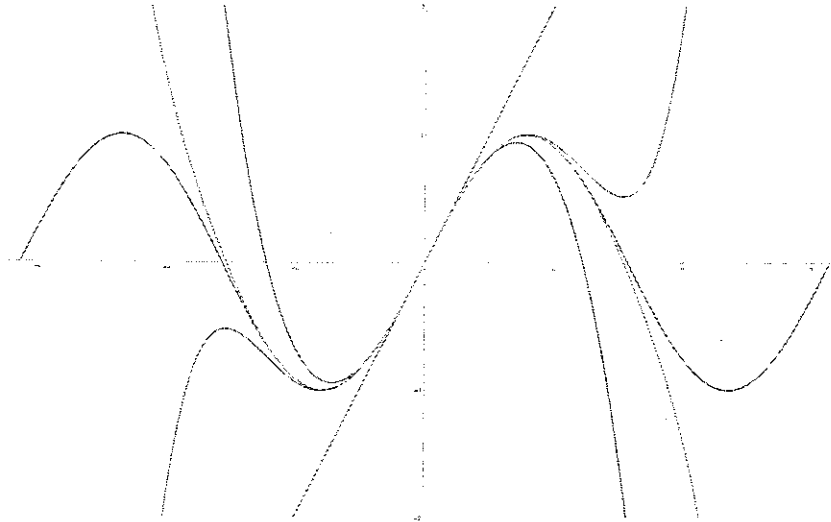
1.  $f(x) = e^x$  as well as  $T_1(x)$ ,  $T_2(x)$ ,  $T_3(x)$ ,  $T_4(x)$  and  $T_5(x)$  are shown:



3.  $f(x) = \cos(x)$  as well as  $T_1(x)$ ,  $T_2(x)$ ,  $T_4(x)$ ,  $T_6(x)$ , and  $T_8(x)$  are shown:



2.  $f(x) = \sin(x)$  as well as  $T_1(x)$ ,  $T_3(x)$ ,  $T_5(x)$ , and  $T_7(x)$  are shown:



Now consider  $f(x) = \frac{1}{1-x}$  based at 0.

Find the 10<sup>th</sup> Taylor polynomial.

What is the error bound on  $[-1/2, 1/2]$ ?

What is the error bound on  $[-2, 2]$ ?

$$f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$$

$$f'(x) = -(1-x)^{-2} \Rightarrow f'(0) = 1$$

$$f''(x) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$$

$$f'''(x) = 2 \cdot 3(1-x)^{-4} \Rightarrow f'''(0) = 3!$$

$$f^{(4)}(x) = 2 \cdot 3 \cdot 4(1-x)^{-5} \Rightarrow f^{(4)}(0) = 4!$$

etc. ...

$$T_{10}(x) = 1 + \frac{1}{1!} 1 x^1 + \frac{1}{2!} 2 x^2 + \frac{1}{3!} 3! x^3 + \frac{1}{4!} 4! x^4 + \dots + \frac{1}{10!} 10! x^{10}$$
$$= 1 + x + x^2 + x^3 + \dots + x^{10}$$

$$f^{(11)}(x) = 11! (1-x)^{-12}$$
$$= \frac{11!}{(1-x)^{12}} \leq M$$

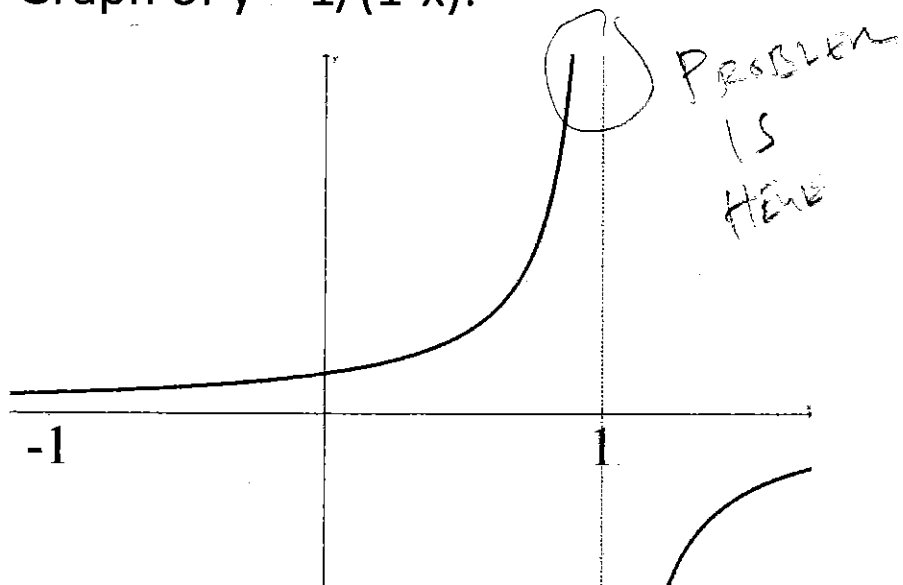
???

$\left[-\frac{1}{2}, \frac{1}{2}\right] \Rightarrow$

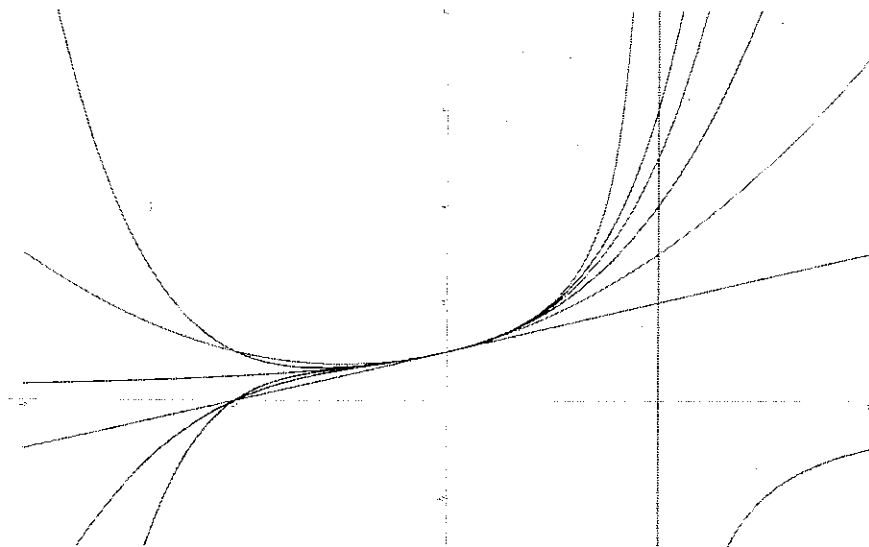
$$M = \frac{11!}{\left(\frac{1}{2}\right)^{12}}$$

$[-2, 2] \Rightarrow$  NO BOUND!!!  
INFINITY  
(ASYMPTOTE AT  $x=1$ )

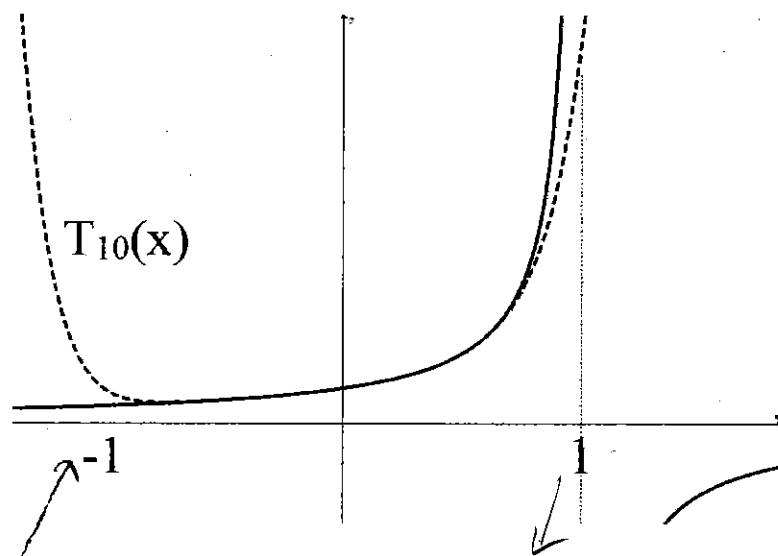
Graph of  $y = 1/(1-x)$ :



$f(x) = \frac{1}{1-x}$  as well as  $T_1(x)$ ,  $T_2(x)$ ,  $T_3(x)$ ,  $T_4(x)$ , and  $T_5(x)$  are shown:



Graph of  $f(x) = \frac{1}{1-x}$  and  $T_{10}(x)$ :



EVEN THOUGH THERE IS NO ASYMPTOTE HERE

SYMMETRY CAUSES THE TAYLOR POLYNOMIAL BASED AT  $b=0$  TO BE "STEEP" HERE AS WELL (ONE AWAY ON BOTH SIDES)

By Friday, we discuss all the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\Rightarrow \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$$

$$\Rightarrow -\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1}$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots$$

$$\Rightarrow \arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

} NEXT  
DNE

The open interval of convergence for all three of these series:  $-1 < x < 1$ .

## Sigma Notation Notes

Definition:

$$\sum_{k=a}^b f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b-1) + f(b)$$

You try: Expand these

$$\sum_{i=1}^3 \frac{(-1)^i}{i^2} x^i = \frac{(-1)^1}{1^2} x^1 + \frac{(-1)^2}{2^2} x^2 + \frac{(-1)^3}{3^2} x^3 = -x + \frac{1}{4}x^2 - \frac{1}{9}x^3$$

SAME!!!! DIFFERENT WAY TO SUMMARIZE THE SAME PATTERN!

$$\sum_{k=13}^{15} \frac{(-1)^{(k-12)}}{(k-12)^2} x^{k-12} = \frac{(-1)^1}{1^2} x^1 + \frac{(-1)^2}{2^2} x^2 + \frac{(-1)^3}{3^2} x^3 = -x + \frac{1}{4}x^2 - \frac{1}{9}x^3$$

(13-12)                      (15-12)

*Note:* In the examples, above  $i$  and  $k$  are *dummy* variables, used to summarize a pattern.



Constants and adding:

Expand then combine

$$\begin{aligned} & 5 \sum_{k=2}^4 k^2 x^k - 6 \sum_{k=2}^4 \frac{1}{k!} x^k \\ &= 5 \left( (2)^2 x^2 + (3)^2 x^3 + (4)^2 x^4 \right) - 6 \left( \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \right) \\ &= 5 \cdot (2)^2 x^2 + 5(3)^2 x^3 + 5(4)^2 x^4 - \frac{6}{2!} x^2 - \frac{6}{3!} x^3 - \frac{6}{4!} x^4 \\ &= \left( 5 \cdot (2)^2 - \frac{6}{2!} \right) x^2 + \left( 5(3)^2 - \frac{6}{3!} \right) x^3 + \left( 5(4)^2 - \frac{6}{4!} \right) x^4 \end{aligned}$$

SAME AS

$$\sum_{k=2}^4 \left( 5k^2 x^k - \frac{6}{k!} x^k \right) = \sum_{k=2}^4 \underbrace{\left( 5k^2 - \frac{6}{k!} \right)}_{a_k} x^k$$

*Summary:* For adding/subtracting and constant multiples, you can manipulate in the same way you learned to manipulate integrals.

## Derivatives and Integrals

Recall:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \quad \frac{d}{dx}(x^n) = nx^{n-1}$$

Thus,

To differentiate a Taylor series  $\rightarrow$  change  $x^k$  to  $kx^{k-1}$

To integrate a Taylor series  $\rightarrow$  change  $x^k$  to  $\frac{1}{k+1} x^{k+1}$

*Example:* Find the derivative and general antiderivative of

$$f(x) = -x + \frac{1}{8}x^2 - \frac{1}{27}x^3 + \frac{1}{64}x^4 - \frac{1}{125}x^5 = \sum_{k=1}^5 \frac{(-1)^k}{k^3} x^k$$

$$f'(x) = -1 + \frac{1}{4}x - \frac{1}{9}x^2 + \frac{1}{16}x^3 - \frac{1}{25}x^4 = \sum_{k=1}^5 \frac{(-1)^k}{k^3} k x^{k-1}$$

$$f'(x) = \sum_{k=1}^5 \frac{(-1)^k}{k^2} x^{k-1}$$

$$\int f(x) dx = C - \frac{1}{2}x^2 + \frac{1}{3} \frac{1}{8}x^3 - \frac{1}{4} \frac{1}{27}x^4 + \frac{1}{5} \frac{1}{64}x^5 - \frac{1}{6} \frac{1}{125}x^6$$

$$\int f(x) dx = C + \sum_{k=1}^5 \frac{(-1)^k}{k^3} \frac{1}{(k+1)} x^{k+1}$$